(Use only axioms I and II - and their consequences, e.g. MI)

1. Let $\emptyset \neq X \subseteq \mathbb{R}$ and $\bar{u} \in \mathbb{R}$. Define what is meant by that $\bar{u} = \sup X$, that is u is the smallest upper bound of X by filling the blanks below

(i) $x \leqslant \bar{u}$ for .... X;

(ii) if $\mathbb{R} \ni w < \bar{u}$ then w ... for .... X.

State the negation (i.e. $\bar{u} \neq \sup X$).

(the def. of $\sup X$ is already provided here & we have not used III).

2. Do Q1 similarly for $\inf X$ (= the greatest lower bound of X)

3^*. Without assuming III, show that

$$-\sup X = \inf(-X)$$

provided that either $\sup X$ exists in $\mathbb{R}$ or $\inf(-X)$ exists in $\mathbb{R}$.

4^*. Let $\emptyset \neq A, B \subseteq \mathbb{R}$ and

$$A+B := \{a+b : a \in A, b \in B\}$$

Show that

$$\sup(A+B) = \sup A + \sup B$$

provided that $\sup A$ and $\sup B$ exist in $\mathbb{R}$.

5.* Let $f, g : D \to \mathbb{R}$ be functions such that $\sup\{f(x)+g(x) : x \in D\}$
$\sup\{f(x) : x \in D\}$, and $\sup\{g(x) : x \in D\}$
exist in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) : x \in D\} \leq \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}.$$

and provide a counter-example showing that "$\leq$" cannot be replaced by "$=$".

6. Let $a, b, x_1 > 0$ (each positive), and

$$x_{n+1} := x_n + \frac{1}{x_n} \quad \forall n \in \mathbb{N}.$$

Show :

(i) $a^2 < b^2$ iff (= if and only if) $a < b$.

(ii) $x_{n+1} > x_n$ and $x_{n+1}^2 > x_n^2$, $\forall n \in \mathbb{N}$.

(iii) $x_{n+1}^2 \geq x_{n+1} \cdot x_n = x_n^2 + 1$ and $x_{n+1}^2 > n$, $\forall n \in \mathbb{N}$.

(iv) sequences (x_n^2) and (x_n) are not bounded.

(for this last ~~part~~ part we do need the Archimedean property in III)

7. Using the combinatory formular

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (k \leq n, \text{ natural nos.}),$$

Show the Binomial Theorem and Bernoulli Inequality for $a, b > 0$ ($\forall\ k, n \in \mathbb{N}$ with $k \leq n$) :

(i) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

(ii) $(1+a)^n \geq \frac{n(n-1)\cdots(n-k+1)}{k!} a^k$.

8*. "Solve" the inequality system :

(#) $4 < |x+2| + |x-1| \leqslant 5$,

that is, let X consist of all x satisfying the above inequalities, concretely express X.

Hint : Try to remove the absolute value signs separately in each of the following cases :

(i) both $(x+2)$ and $(x-1)$ are $\geqslant 0$,

(ii) both . — - . are $\leqslant 0$,

(iii) $x+2 \geqslant 0$ but $(x-1) \leqslant 0$,

(iv) . . . .

Let X_1 consist of all x satisfying (i) and (#).

Show $X_1 = (\frac{3}{2}, 2]$. Similarly $X_4 = \emptyset$, $X_3 = \emptyset$ and $X_2 = [-3, -\frac{5}{2})$.

Note. You may also divide into 3 cases.

e.g.

$\mathbb{R} = (-\infty, -2) \cup [-2, 1] \cup (1, +\infty)$

$= I_1 \cup I_2 \cup I_3$ where

$I_1 := (-\infty, -2)$

$I_2 := [-2, 1]$

$I_3 := (1, \infty)$

(or $\mathbb{R} = (-\infty, -2] \cup (-2, 1) \cup [1, +\infty)$

甚至亦可 $\mathbb{R} = (-\infty, -2] \cup [-2, 1] \cup [1, +\infty)$)

Then $X = (X \cap I_1) \cup (X \cap I_2) \cup (X \cap I_3)$

Further

$$X \cap I_1 = \{x \in \mathbb{R} : 4 < -(x+2) + (1-x) \leq 5\}$$
$$\overset{?}{=} \left[-3, -\tfrac{5}{2}\right)$$

$$X \cap I_2 = \{x \in \mathbb{R} : 4 < (x+2) + (1-x) \leq 5\}$$
$$\overset{?}{=} \emptyset$$

$$X \cap I_3 = \{x \in \mathbb{R} : 4 < (x+2) + (x-1) \leq 5\}$$
$$\overset{?}{=} \left(\tfrac{3}{2}, 2\right].$$

$$\therefore X = \left[-3, -\tfrac{5}{2}\right) \cup \left(\tfrac{3}{2}, 2\right]$$